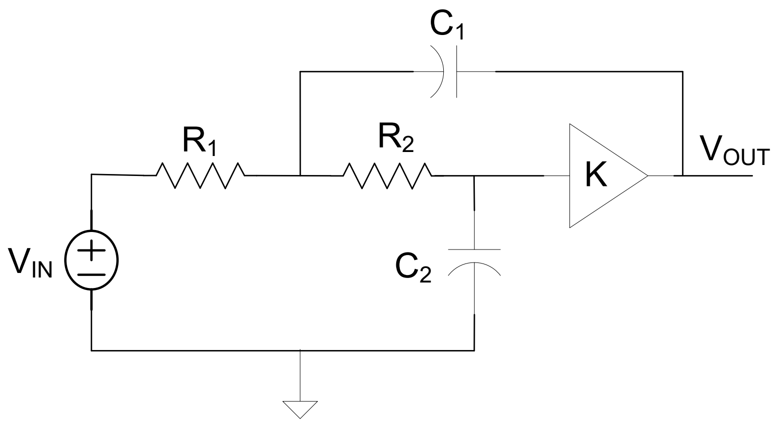
EE 508
Lecture 19

Sensitivity Functions

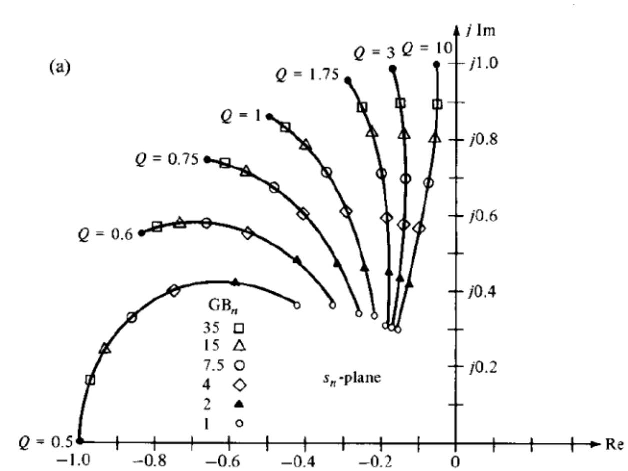
- Comparison of Filter Structures
- Performance Prediction

Review from last time

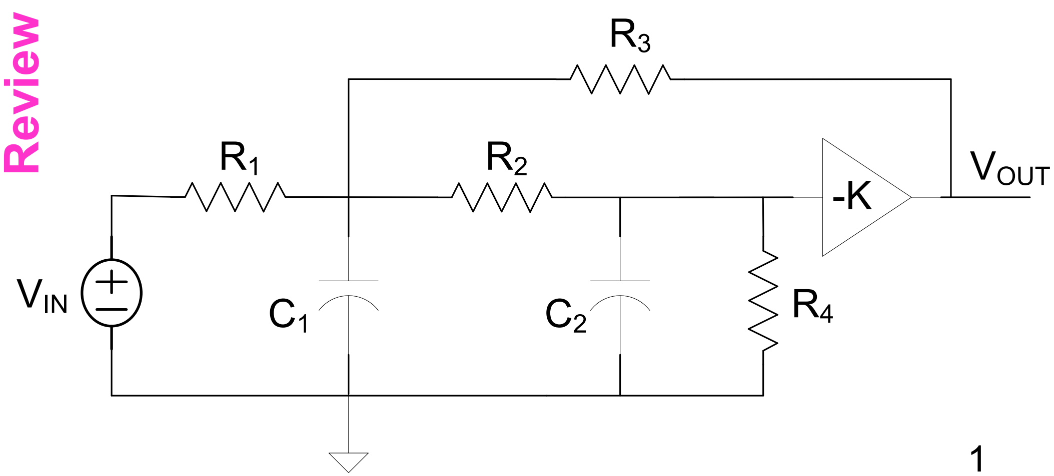
What causes the dramatic differences in performance between these two structures?
How can the performance of different structures be compared in general?

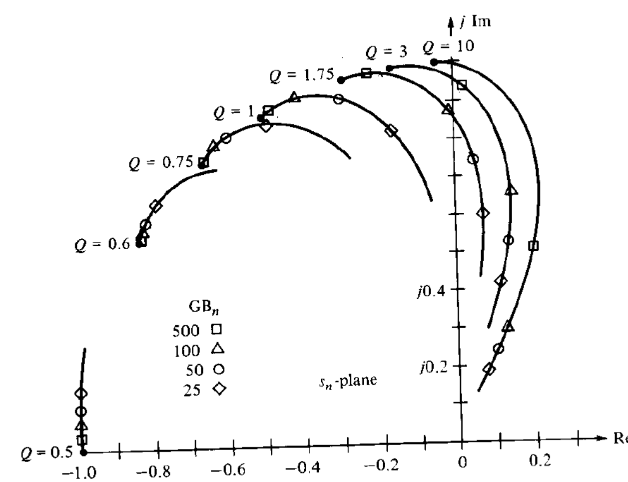


$$T(s)=K\frac{\frac{1}{R_1R_2C_1C_2}}{s^2+s\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1-K}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}}$$



Equal R, Equal C, Q=10 Pole Locus vs GB_N

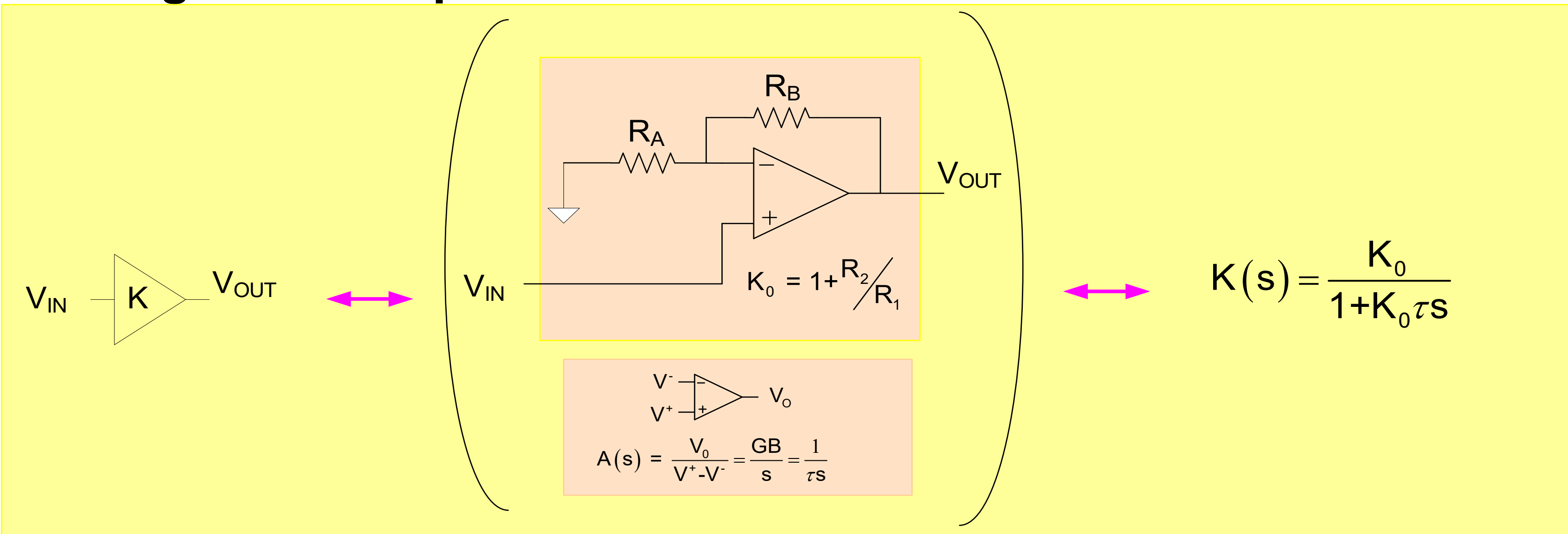


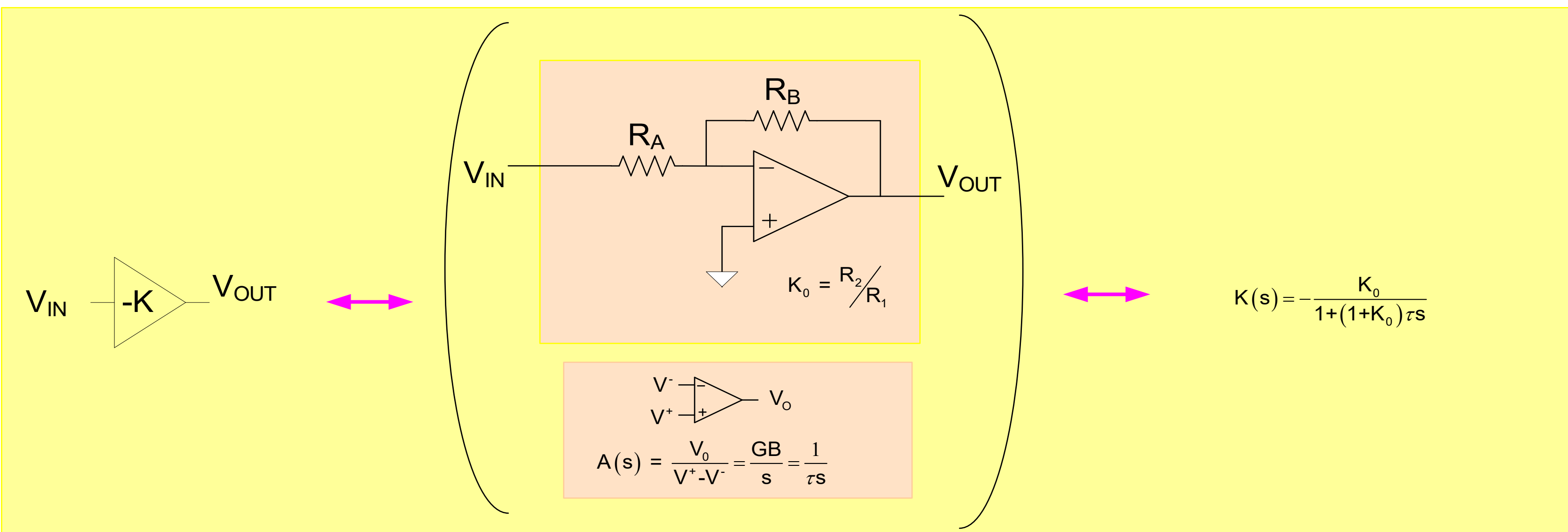


$$T(s)=-K\frac{\frac{1}{R_1R_2C_1C_2}}{s^2+s\left[\frac{1}{R_1C_1}\left(1+\frac{R_1}{R_3}\right)+\frac{1}{R_4C_2}+\frac{1}{R_2C_2}\left(1+\frac{C_2}{C_1}\right)\right]+\left[\frac{1+(R_1/R_3)(1+K)+(R_1/R_4)\left(1+(R_2/R_3)+(R_2/R_1)\right)}{R_1R_2C_1C_2}\right]}$$

Review from last time

Modeling of the Amplifiers



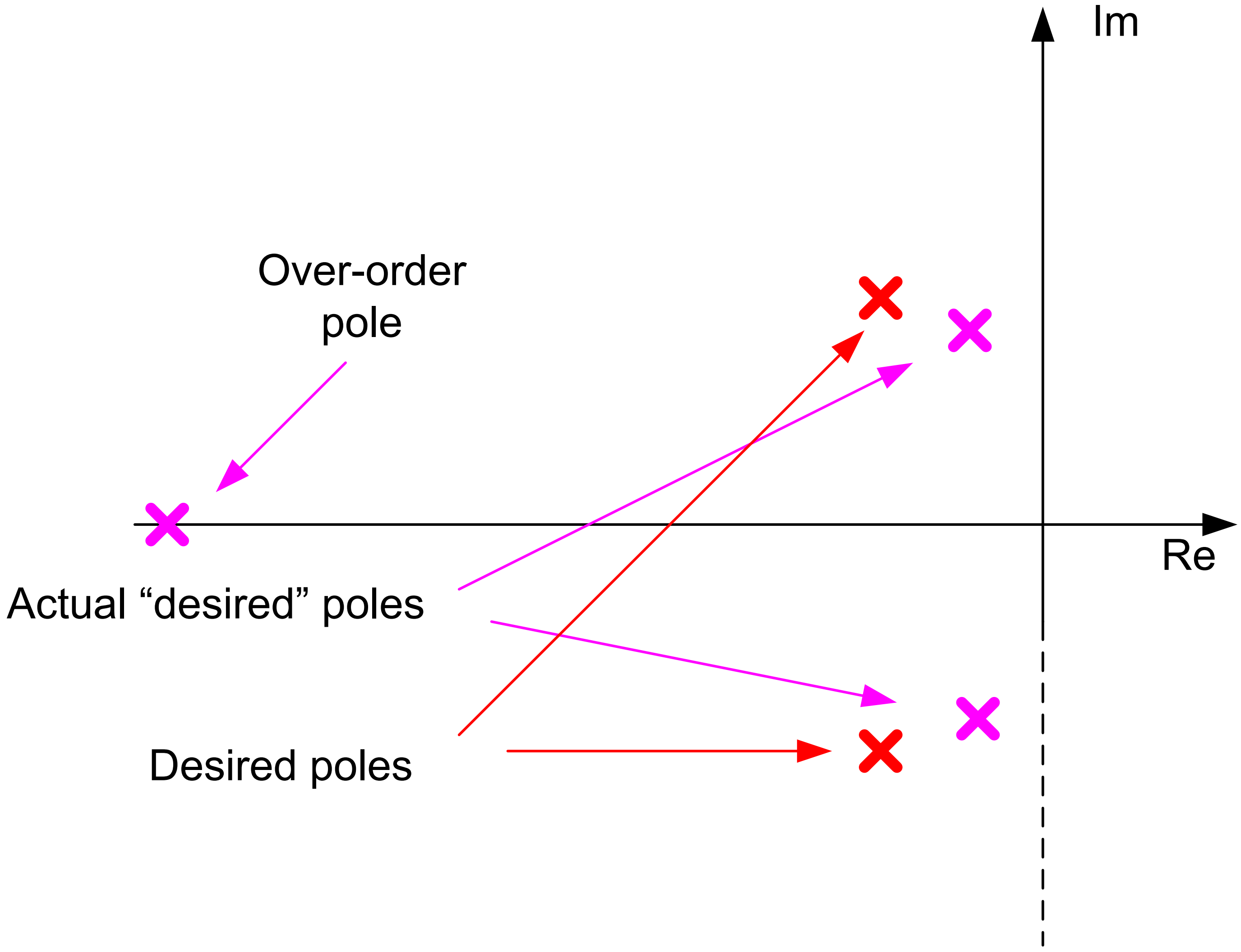


Different implementations of the amplifiers are possible

Have used the op amp time constant in these models $\tau = GB^{-1}$

Review from last time

Effects of GB on poles of KRC and -KRC Lowpass Filters



GB effects in KRC and -KRC Lowpass Filter

$$T(s)=\frac{K_0\omega_0^2}{s^2+s\left[\frac{\omega_0}{Q}\right]+\omega_0^2+K_0\tau s\left(s^2+s\left[\frac{\omega_0}{Q}\left(1+K_0Q\sqrt{\frac{R_1C_1}{R_2C_2}}\right)\right]+\omega_0^2\right)}$$

$$T(s)=-K_0\frac{\frac{1}{R_1R_2C_1C_2}}{\left(s^2+s\left[\frac{1}{R_1C_1}\left(1+\frac{R_1}{R_3}\right)+\frac{1}{R_4C_2}+\frac{1}{R_2C_2}\left(1+\frac{C_2}{C_1}\right)\right]+\left[\frac{1+(R_1/R_3)(1+K_0)+(R_1/R_4)\big(1+(R_2/R_3)+(R_2/R_1)\big)}{R_1R_2C_1C_2}\right]\right)+\tau s(1+K_0)\left(s^2+s\left[\frac{1}{R_1C_1}\left(1+\frac{R_1}{R_3}\right)+\frac{1}{R_4C_2}+\frac{1}{R_2C_2}\left(1+\frac{C_2}{C_1}\right)\right]+\left[\frac{1+(R_1/R_3)+(R_1/R_4)\big(1+(R_2/R_3)+(R_2/R_1)\big)}{R_1R_2C_1C_2}\right]\right)}$$

- Analytical expressions for ω_0, Q, poles, zeros, and other key parameters are unwieldly in these circuits and as bad or worse in many other circuits (require solution of 3rd order polynomial!!)
- Sensitivity metrics give considerable insight into how filters perform and are widely used to assess relative performance
- Need sensitivity characterization of real numbers as well as complex quantities such as poles and zeros
- If sensitivity expressions are obtained for a given structure, it can be catalogued rather than recalculated
- **Since analytical expressions for key parameters are unwieldly in even simple circuits, obtaining expressions for the purpose of calculating sensitivity appears to be a formidable task !**

Sensitivity Characterization of Filter Structures

Let F be a filter characteristic of interest

F might be ω_0 or Q of a pole or zero, a band edge, a peak frequency, a BW, T(s), |T(jω)|, a coefficient in T(s), etc

Can express F in terms of all components and model parameters as

$$F=f(R_1,\ldots R_{k1},C_1,\ldots C_{k2},L_{l1},\ldots L_{lk3},\tau_1,\ldots \tau_{k4},W_1,\ldots W_{k5},L_1,\ldots L_{k5},\ldots)$$

$$F=f(x_1,x_2,\ \ldots x_k)$$

The differential dF of the multivariate function F can be expressed as

$$dF=\frac{\partial F}{\partial R_1}dR_1+\frac{\partial F}{\partial R_2}dR_2+\ldots.+\frac{\partial F}{\partial R_{k1}}dR_{k1}$$
$$+\frac{\partial F}{\partial C_1}dC_1+\frac{\partial F}{\partial C_2}dC_2+\ldots.+\frac{\partial F}{\partial C_{k2}}dRC_{k2}$$
$$+\ \ldots\ldots..$$

$$dF=\sum_{i=1}^{k}\frac{\partial F}{\partial x_i}dx_i$$

Define the standard sensitivity function as

$$\mathrm{S}_x^f = \frac{\partial f}{\partial x} \bullet \frac{x}{f}$$

S_x^f Is widely used except when x or f assume extreme values of 0 or ∞

Define the derivative sensitivity function as

$$\mathcal{S}_x^f = \frac{\partial f}{\partial x}$$

$\mathcal{S}_x^f$ Is more useful when x or f ideally assume extreme values of 0 or ∞

Consider the normalized differential $\dfrac{dF}{F}$

$$\frac{dF}{F} \cong \frac{\Delta F}{F}$$

This approximates the relative (percent if multiply by 100) change in F due to changes in ALL components

$$\frac{dF}{F} = \frac{\displaystyle\sum_{i=1}^{k} \frac{\partial F}{\partial x_i} dx_i}{F} = \sum_{i=1}^{k} \frac{\partial F}{\partial x_i} \bullet \frac{dx_i}{F} \overset{\text{All } x_i \neq 0,\infty}{=} \sum_{i=1}^{k} \left(\frac{\partial F}{\partial x_i} \bullet \frac{x_i}{F} \right) \bullet \frac{dx_i}{x_i}$$

This can be expressed in terms of the standard sensitivity function as

$$\frac{dF}{F} \overset{\text{All } x_i \neq 0,\infty}{=} \sum_{i=1}^{k} \left(S_{x_i}^{f} \bullet \frac{dx_i}{x_i} \right)$$

This relates the relative (percent if multiply by 100) change in F to the sensitivity function and the relative (percent if multiply by 100) change in each component

Consider the normalized differential

$$\frac{dF}{F}=\sum_{i=1}^{k}\left(S_{x_i}^{f}\bullet\frac{dx_i}{x_i}\right)$$

This can be expressed as

$$\frac{dF}{F}=\left(\sum_{all\ resistors} S_{R_i}^{f}\bullet\frac{dR_i}{R_i}\right)+\left(\sum_{all\ capacitors} S_{C_i}^{f}\bullet\frac{dC_i}{C_i}\right)+\left(\sum_{all\ opamps} S_{\tau_i}^{f}\bullet\frac{d\tau_i}{\tau_i}\right)+\ldots$$

Often interested in $\dfrac{dF}{F}$ evaluated at the ideal (or nominal value)

If the nominal values are all not extreme (0 or ∞), then

$$\frac{dF}{F}=\sum_{i=1}^{k}\left(S_{x_i}^{f}\Big|_{\vec{X}_N}\bullet\frac{dx_i}{x_{iN}}\right)$$

The normalized differential – a different perspective

$$\frac{dF}{F} = \sum_{i=1}^{k}\left(S_{x_i}^{f}\Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}} \right)$$

Consider the multivariate Taylors series expansion of F

$$F(\vec{X}) = F(\vec{X}_N) + \sum_{i=1}^{k} \frac{\partial F}{\partial x_i}\Big|_{\vec{X}_N} (x_i - x_{iN}) + \left[\frac{1}{2!}\sum_{i=1}^{k} \frac{\partial^2 F}{\partial x_i^2}\Big|_{\vec{X}_N} (x_i - x_{iN})^2 + \sum_{\substack{i=1\\ j=1\\ i\neq j}}^{k,k} \frac{\partial^2 F}{\partial x_i \partial x_j}\Big|_{\vec{X}_N} (x_i - x_{iN})(x_j - x_{jN}) \right] + \ldots..$$

$$F(\vec{X}) \cong F(\vec{X}_N) + \sum_{i=1}^{k} \frac{\partial F}{\partial x_i}\Big|_{\vec{X}_N} (x_i - x_{iN})$$

$$F(\vec{X}) - F(\vec{X}_N) \cong \sum_{i=1}^{k} \frac{\partial F}{\partial x_i}\Big|_{\vec{X}_N} (x_i - x_{iN})$$

$$\Delta F(\vec{X}) \cong \sum_{i=1}^{k} \frac{\partial F}{\partial x_i}\Big|_{\vec{X}_N} \Delta x_i$$

The normalized differential – a different perspective

$$\frac{dF}{F}=\sum_{i=1}^{k}\left(S_{x_i}^{f}\Big|_{\vec{X}_N}\bullet\frac{dx_i}{x_{iN}}\right)$$

Consider the multivariate Taylors series expansion of F

$$\Delta F\left(\vec{X}\right)\cong\sum_{i=1}^{k}\frac{\partial F}{\partial x_i}\bigg|_{\vec{X}_N}\Delta x_i$$

$$\frac{\Delta F\left(\vec{X}\right)}{F}\cong\sum_{i=1}^{k}\frac{\partial F}{\partial x_i}\bigg|_{\vec{X}_N}\frac{\Delta x_i}{F}=\sum_{i=1}^{k}\frac{\partial F}{\partial x_i}\bigg|_{\vec{X}_N}\frac{x_i}{x_i}\frac{\Delta x_i}{F}=\sum_{i=1}^{k}\left(\frac{\partial F}{\partial x_i}\bigg|_{\vec{X}_N}\frac{x_i}{F}\right)\frac{\Delta x_i}{x_i}$$

$$\frac{\Delta F}{F}\cong\sum_{i=1}^{k}\left(S_{x_i}^{f}\Big|_{\vec{X}_N}\right)\frac{\Delta x_i}{x_i}$$

Note this is essentially the same expression that was arrived at from the sensitivity analysis approach

$$\frac{dF}{F} = \sum_{i=1}^{k} \left(\boxed{S_{x_i}^{f} \Big|_{\vec{X}_N}} \bullet \boxed{\frac{dx_i}{x_{iN}}} \right)$$

Dependent only on components (not circuit structure)

Dependent on circuit structure **(for some circuits, also not dependent on components)**

The sensitivity functions are thus useful for comparing different circuit structures

The variability which is the product of the sensitivity function and the normalized component differential is more important for predicting circuit performance

Variability Formulation

$$V_{x_i}^f = S_{x_i}^f \Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}}$$

$$\frac{dF}{F} = \sum_{i=1}^{k} V_{x_i}^f \Big|_{\vec{X}_N}$$

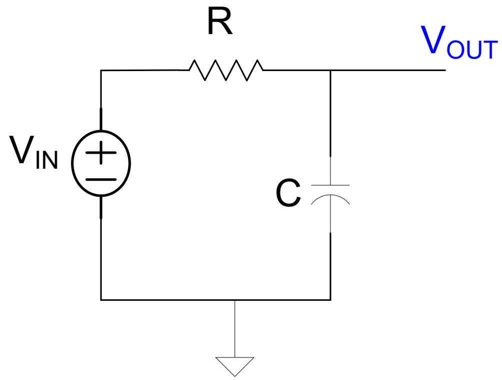
Variability includes effects of both circuit structure and components on performance

Often interested in circuits whose performance is not affected by changes in component values. In such cases:

If component variations are small, high sensitivities are acceptable

If component variations are large, low sensitivities are usually critical

Example



$$T(s) = \frac{1}{1+RCs} = \frac{\omega_0}{s+\omega_0}$$

If $\omega_0 = 1/RC$, determine $S_R^{\omega_0}$ and $S_C^{\omega_0}$

$$S_R^{\omega_0} = \frac{\partial \omega_0}{\partial R} \bullet \frac{R}{\omega_0}$$

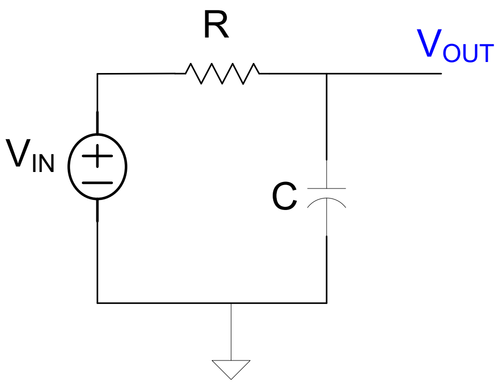
$$S_R^{\omega_0} = \left(\frac{-1}{R^2 C}\right) \bullet \frac{R}{\omega_0}$$

$$S_R^{\omega_0} = -\frac{1}{R}\left(\frac{1}{RC}\right) \bullet \frac{R}{\omega_0} = -\frac{1}{R}(\omega_0) \bullet \frac{R}{\omega_0} = -1$$

Likewise

$$S_C^{\omega_0} = -1$$

Example



$$T(s) = \frac{1}{1+RCs} = \frac{\omega_0}{s+\omega_0}$$

$$\omega_0 = \frac{1}{RC}$$

$$\frac{d\omega_0}{\omega_0} = \sum_{i=1}^{k} V_{x_i}^{\omega_0}\Big|_{\vec{X}_N}$$

$$V_{x_i}^{f} = S_{x_i}^{f}\Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}}$$

$$S_R^{\omega_0} = -1 \qquad S_C^{\omega_0} = -1$$

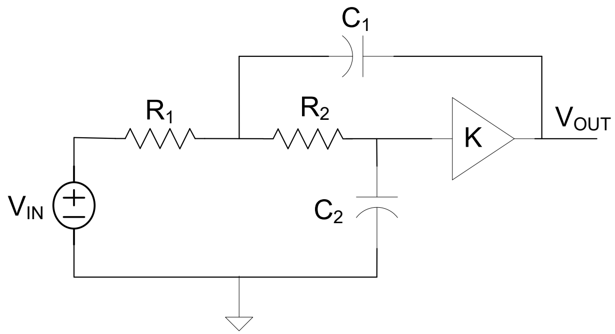
Thus a 1% increase in R will cause approximately a 1% decrease in ω_0

a 1% increase in C will cause approximately a 1% decrease in ω_0

a 1% increase in both C and R will cause approximately a 2% decrease in ω_0

Example

$$T(s)=K\frac{\frac{1}{R_1R_2C_1C_2}}{s^2+s\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1-K}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}}$$



$$Q=\frac{1}{\sqrt{\frac{R_2C_2}{R_1C_1}}+\sqrt{\frac{R_1C_2}{R_2C_1}}+(1-K)\sqrt{\frac{R_1C_1}{R_2C_2}}}$$

$$\omega_0=\frac{1}{\sqrt{R_1R_2C_1C_2}}$$

Determine $\quad S_{C_1}^{\omega_0}\quad S_{C_2}^{\omega_0}\quad S_{R_1}^{\omega_0}\quad S_{R_2}^{\omega_0}$

$$S_{C_1}^{\omega_0}=\frac{\partial\left[\frac{1}{\sqrt{R_1R_2C_1C_2}}\right]}{\partial C_1}\frac{C_1}{\omega_0}$$

$$S_{C_1}^{\omega_0}=\frac{1}{\sqrt{R_1R_2C_2}}\frac{\partial\left[\frac{1}{\sqrt{C_1}}\right]}{\partial C_1}\frac{C_1}{\omega_0}$$

$$S_{C_1}^{\omega_0}=\frac{1}{\sqrt{R_1R_2C_2}}\left(-\tfrac{1}{2}C_1^{-3/2}\right)\frac{C_1}{\omega_0}$$

$$S_{C_1}^{\omega_0}=-\tfrac{1}{2}\frac{1}{\sqrt{R_1R_2C_2}}\left(\frac{1}{\sqrt{C_1}C_1}\right)\frac{C_1}{\omega_0}$$

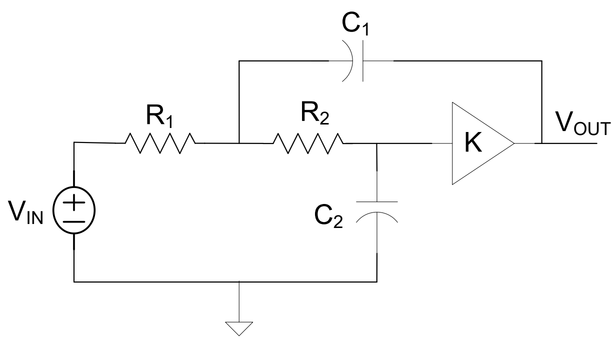
$$S_{C_1}^{\omega_0}=-\tfrac{1}{2}\frac{1}{\sqrt{R_1R_2C_2C_1}}\left(\frac{1}{C_1}\right)\frac{C_1}{\omega_0}$$

$$S_{C_1}^{\omega_0}=-\tfrac{1}{2}\omega_0\left(\frac{1}{C_1}\right)\frac{C_1}{\omega_0}$$

$$S_{C_1}^{\omega_0}=-\tfrac{1}{2}$$

Example

$$T(s)=K\frac{\frac{1}{R_1R_2C_1C_2}}{s^2+s\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1-K}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}}$$



$$Q=\frac{1}{\sqrt{\frac{R_2C_2}{R_1C_1}}+\sqrt{\frac{R_1C_2}{R_2C_1}}+(1-K)\sqrt{\frac{R_1C_1}{R_2C_2}}}$$

$$\omega_0=\frac{1}{\sqrt{R_1R_2C_1C_2}}$$

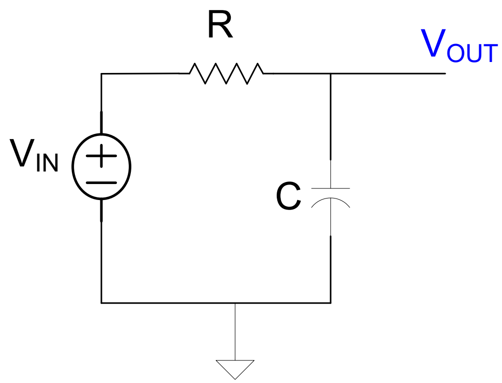
Determine $S_{C_1}^{\omega_0}$ $S_{C_2}^{\omega_0}$ $S_{R_1}^{\omega_0}$ $S_{R_2}^{\omega_0}$

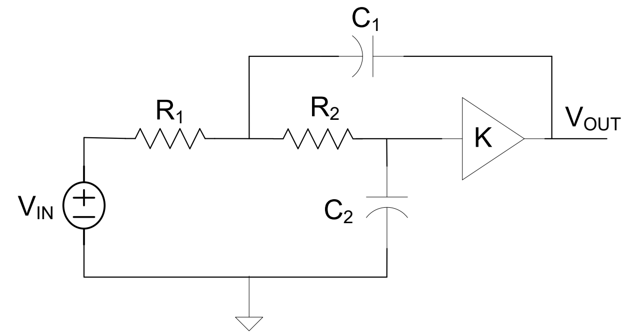
$$S_{C_1}^{\omega_0}=-\frac{1}{2}$$

Likewise

$$S_{C_2}^{\omega_0}=S_{R_1}^{\omega_0}=S_{R_2}^{\omega_0}=-\frac{1}{2}$$

Observation:





$$\omega_0 = \frac{1}{\sqrt{R_1R_2C_1C_2}}$$

$$\omega_0 = {}^{1}\!/_{RC}$$

$$S_R^{\omega_0} = -1 \qquad S_C^{\omega_0} = -1$$

$$\sum_{\text{All resistors}} S_{R_i}^{\omega_0} = -1 \qquad \sum_{\text{All capacitors}} S_{C_i}^{\omega_0} = -1$$

$$S_{R_1}^{\omega_0} = -{}^{1}\!/_{2} \qquad S_{C_1}^{\omega_0} = -{}^{1}\!/_{2}$$

$$S_{R_2}^{\omega_0} = -{}^{1}\!/_{2} \qquad S_{C_2}^{\omega_0} = -{}^{1}\!/_{2}$$

$$\sum_{\text{All resistors}} S_{R_i}^{\omega_0} = -1 \qquad \sum_{\text{All capacitors}} S_{C_i}^{\omega_0} = -1$$

At this stage, this is just an observation about summed sensitivities but later will establish some fundamental properties of summed sensitivities

Consider

$$\frac{dF}{F}=\left(\sum_{all\ resistors} S^{f}_{R_i}\bullet\frac{dR_i}{R_i}\right)+\left(\sum_{all\ capacitors} S^{f}_{C_i}\bullet\frac{dC_i}{C_i}\right)+\left(\sum_{all\ opamps} S^{f}_{\tau_i}\bullet\frac{d\tau_i}{\tau_i}\right)+\ldots$$

The nominal value of the time constant of the op amps is 0 so this expression can not be evaluated at the ideal (nominal) value of GB=∞ (equivalently τ=0)

Let $\{x_i\}$ be the components in a circuit whose nominal value is not 0

Let $\{y_i\}$ be the components in a circuit whose nominal value is 0

$$\frac{dF}{F}=\sum_{i=1}^{kx}\frac{\partial F}{\partial x_i}\bullet\frac{dx_i}{F}+\sum_{i=1}^{ky}\frac{\partial F}{\partial y_i}\bullet\frac{dy_i}{F}=\sum_{i=1}^{k}\left(\frac{\partial F}{\partial x_i}\bullet\frac{x_i}{F}\right)\bullet\frac{dx_i}{x_i}+\frac{1}{F}\sum_{i=1}^{ky}\frac{\partial F}{\partial y_i}dy_i$$

$$\frac{dF}{F}=\sum_{i=1}^{kx}\left(S^{f}_{x_i}\Big|_{\vec{X}_N,\,Y_N=0}\bullet\frac{dx_i}{x_i}\right)+\frac{1}{F_N}\sum_{i=1}^{ky}\left(\mathcal{S}^{f}_{y_i}\Big|_{\vec{X}_N,\,Y_N=0}\bullet y_i\right)$$

This expression can be used for predicting the effects of all components in a circuit

Can set Y_N=0 before calculating $S^{f}_{x_i}$ **functions**

$$\frac{dF}{F}=\sum_{i=1}^{k}\left(S_{x_i}^{f}\Big|_{\vec{X}_N}\bullet\frac{dx_i}{x_i}\right)+\frac{1}{F_N}\sum_{i=1}^{ky}\left(\mathcal{S}_{y_i}^{f}\Big|_{\vec{Y}_N=0}\bullet y_i\right)$$

Low sensitivities in a circuit are often preferred but in some applications, low sensitivities would be totally unacceptable

Examples where low sensitivities are unacceptable are circuits where a charactristics F must be tunable or adjustable!

Some useful sensitivity theorems

$$S_x^{kf} = S_x^f$$ where k is a constant

$$S_x^{f^n} = n \bullet S_x^f$$

$$S_x^{1/f} = -S_x^f$$

$$S_x^{\sqrt{f}} = \frac{1}{2} S_x^f$$

$$S_x^{\prod_{i=1}^{k} f_i} = \sum_{i=1}^{k} S_x^{f_i}$$

Some useful sensitivity theorems (cont)

$$S_x^{f/g} = S_x^f - S_x^g$$

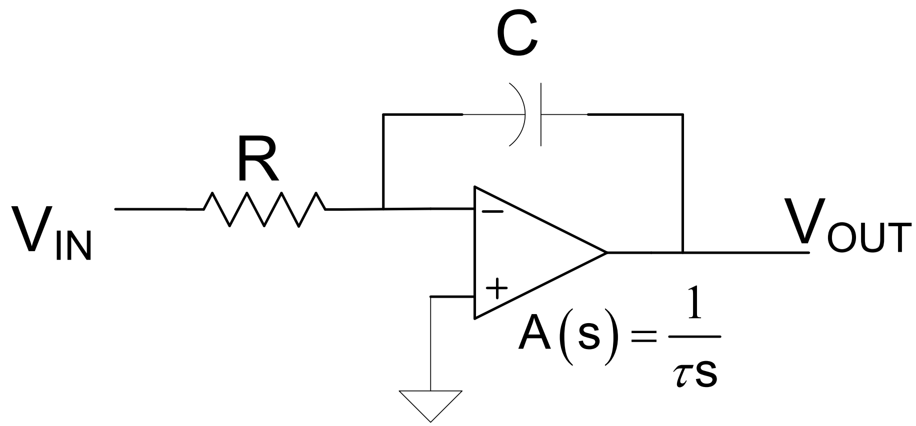
$$S_x^{\sum_{i=1}^{k} f_i} = \frac{\sum_{i=1}^{k} f_i S_x^{f_i}}{\sum_{i=1}^{k} f_i}$$

$$S_{1/x}^{f} = -S_x^f$$

Stay Safe and Stay Healthy !

End of Lecture 19

Example:



Ideally $\quad I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$ $\quad I_0$ termed the unity gain freq of integrator

$$I_0=\frac{1}{RC}$$

I_0 is one of the most important parameters of an integrator used in a filter

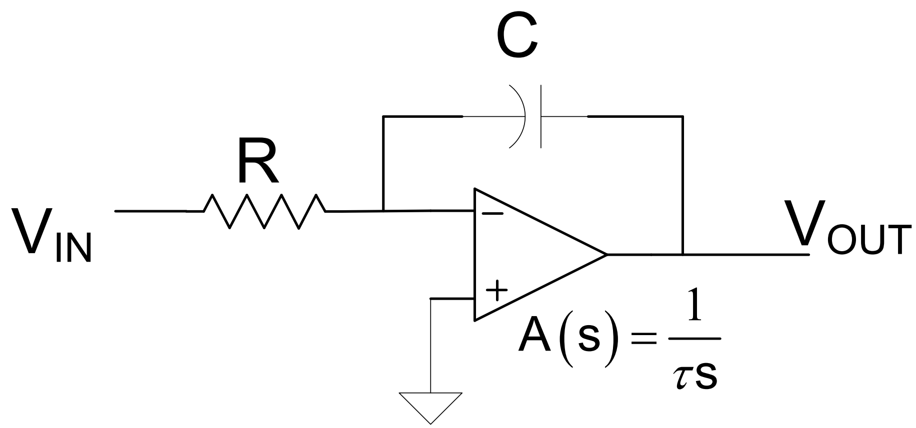
Assume ideally R=1K, C=3.18nF so that I_O=50KHz

Assume actually GB=600KHz, R=1.05K, and C=3.3nF

a) Determine an approximation to the actual unity gain frequency using a sensitivity analysis

b) Write an analytical expression for the actual unity gain frequency

Example:



Assume ideally R=1K, C=3.18nF so that I_O=50KHz

Actually GB=600KHz, R=1.05K, and C=3.3nF

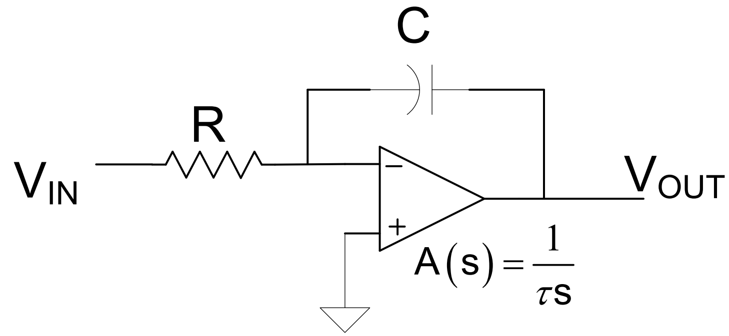
Observe

$$\frac{\Delta R}{R} = \frac{.05K}{1K} = .05$$

$$\frac{\Delta C}{C} = \frac{.12nF}{3.18nF} = .038$$

$$\frac{I_0}{GB} = \tau I_0 = \frac{50KHz}{600KHz} = .083$$

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$$

Solution:

Define I_{0A} to be the actual unity gain frequency

$$I_0 = \frac{1}{RC}$$

$$\frac{dF}{F} = \sum_{i=1}^{k}\left(S_{x_i}^{f}\Big|_{\vec{X}_N,\vec{Y}_N=0} \bullet \frac{dx_i}{x_i}\right) + \frac{1}{F_N}\sum_{i=1}^{ky}\left(\mathcal{S}_{y_i}^{f}\Big|_{\vec{X}_N,\vec{Y}_N=0} \bullet y_i\right)$$

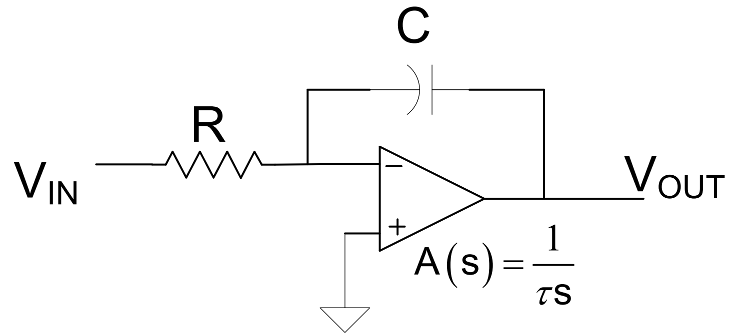
$$\frac{dI_{0A}}{I_{0A}} = \left[S_R^{I_{0A}}\Big|_{R_N,C_N,\tau=0}\right]\frac{dR}{R_N} + \left[S_C^{I_{0A}}\Big|_{R_N,C_N,\tau=0}\right]\frac{dC}{C_N} + \frac{1}{I_{0N}}\left(\mathcal{S}_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0} \bullet \tau\right)$$

$$S_R^{I_{0A}}\Big|_{R_N,C_N,\tau=0} = S_R^{I_0}\Big|_{R_N,C_N} \qquad S_C^{I_{0A}}\Big|_{R_N,C_N,\tau=0} = S_C^{I_0}\Big|_{R_N,C_N}$$

$$S_R^{I_0}\Big|_{R_N,C_N} = -1 \qquad S_C^{I_0}\Big|_{R_N,C_N} = -1$$

It remains to calculate $\mathcal{S}_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0}$

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$$

Solution:

Still need $\mathcal{S}_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0}$

Define I_{0A} to be the actual unity gain frequency

$$I_A(s) = -\frac{1}{RCs+\tau s(1+RCs)}$$

$$I_A(j\omega) = -\frac{1}{-\tau RC\omega^2 + j(\omega RC+\tau\omega)}$$

$$\left|I_A(j\omega)\right|^2 = \frac{1}{(RC)^2\tau^2\omega^4+\omega^2(RC+\tau)^2}$$

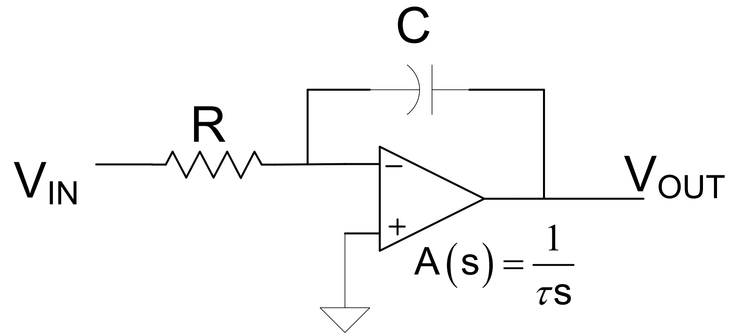
$$\left|I_A(j\omega)\right|^2 = \frac{1}{(RC)^2\tau^2\omega^4+\omega^2(RC+\tau)^2} \underset{defn}{=} 1$$

$$\frac{1}{(RC)^2\tau^2 I_{0A}^4+I_{0A}^2(RC+\tau)^2} = 1$$

$$(RC)^2\tau^2 I_{0A}^4+I_{0A}^2(RC+\tau)^2 = 1$$

$$\mathcal{S}_{\tau}^{I_{0A}}\Big|_{\vec{X}_N,\vec{Y}_N=0} \quad =?$$

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_0}{s}$$

Solution:

Still need $\left.\mathcal{S}_{\tau}^{I_{0A}}\right|_{\vec{X}_N,\vec{Y}_N=0}$ Define I_{0A} to be the actual unity gain frequency

$$(RC)^2\tau^2 I_{0A}^4 + I_{OA}^2(RC+\tau)^2 = 1$$

$$\left.\mathcal{S}_{\tau}^{I_{0A}}\right|_{\vec{X}_N,\vec{Y}_N=0} = \left.\left(\frac{\partial I_{0A}}{\partial\tau}\right)\right|_{\vec{X}_N,\vec{Y}_N=0}$$

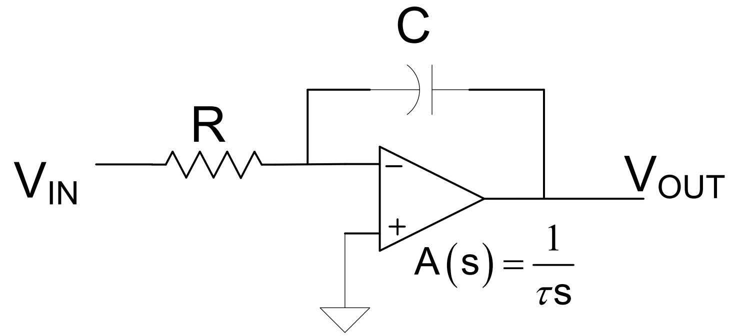
$$(RC)^2\tau^2 4I_{0A}^3\left(\frac{\partial I_{0A}}{\partial\tau}\right) + 2\tau(RC)^2 I_{0A}^4 + 2I_{OA}^1\left(\frac{\partial I_{0A}}{\partial\tau}\right)(RC+\tau)^2 + 2(RC+\tau)I_{OA}^2 = 0$$

Evaluating at $\vec{X}_N,\vec{Y}_N = 0$

$$2I_O^1\left(\left.\frac{\partial I_{0A}}{\partial\tau}\right|_{\vec{X}_N,\vec{Y}_N=0}\right)(RC)^2 + 2(RC)I_O^2 = 0$$

$$\left(\left.\frac{\partial I_{0A}}{\partial\tau}\right|_{\vec{X}_N,\vec{Y}_N=0}\right) = \frac{-I_O}{RC} = \left.\mathcal{S}_{\tau}^{I_{0A}}\right|_{\vec{X}_N,\vec{Y}_N=0} = -I_O^2$$

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_{0N}}{s}$$

Solution:

$$\frac{dI_{0A}}{I_{0A}} = \left[S_R^{I_{0A}} \Big|_{R_N, C_N, \tau=0} \right] \frac{dR}{R_N} + \left[S_C^{I_{0A}} \Big|_{R_N, C_N, \tau=0} \right] \frac{dC}{C_N} + \frac{1}{I_{0N}} \left(\mathcal{S}_\tau^{I_{0A}} \Big|_{\vec{X}_N, \vec{Y}_N=0} \bullet \tau \right)$$

$$S_R^{I_0} \Big|_{R_N, C_N} = S_C^{I_0} \Big|_{R_N, C_N} = -1 \qquad \mathcal{S}_\tau^{I_{0A}} \Big|_{\vec{X}_N, \vec{Y}_N=0} = -I_{ON}^2$$

$$\frac{\Delta R}{R} = .05 \qquad \frac{\Delta C}{C} = .038 \qquad \tau I_0 = .083$$

$$\frac{dI_{0A}}{I_{0A}} = [-1] .05 + [-1] .038 + \frac{1}{I_{0N}} \left(-I_{0N}^2 \bullet \tau \right)$$

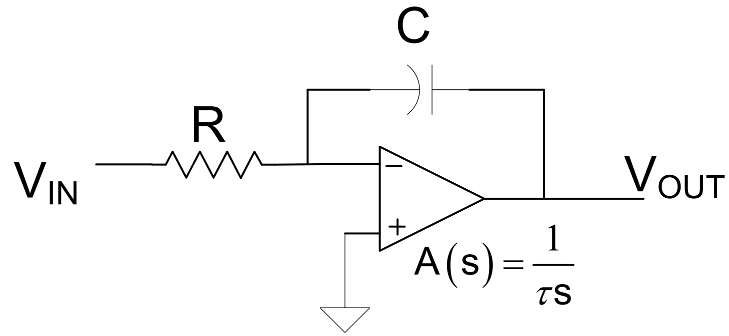
$$\frac{dI_{0A}}{I_{0A}} = [-1] .05 + [-1] .038 + (-.083)$$

Due to passives

$$\frac{dI_{0A}}{I_{0A}} = -.088 - .083$$

Due to actives

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_{0N}}{s}$$

Solution:

$$\frac{dI_{0A}}{I_{0A}} = -.171$$

I_{ON}=50KHz

$$I_{0A} \cong 0.829 I_{0N} = 41.45 KHz$$

Note that with the sensitivity analysis, it was not necessary to ever determine I_{0A} !!

a) Determine an approximation to the actual unity gain frequency using a sensitivity analysis

b) Write an analytical expression for the actual unity gain frequency

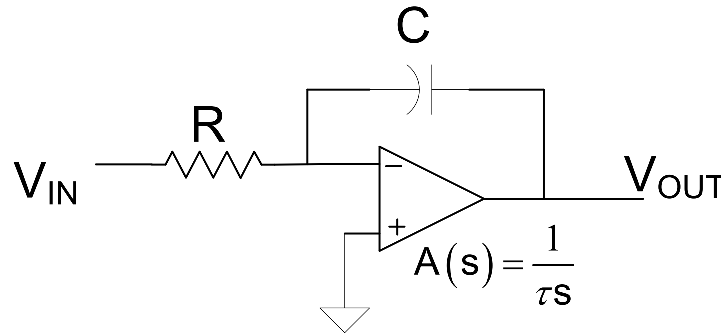
$$(RC)^2 \tau^2 I_{0A}^4 + I_{0A}^2 (RC+\tau)^2 = 1$$

Must solve this quadratic for I_{0A} Solving, obtain I_{OA}=42.6KHz

Note this is close to the value obtained with the sensitivity analysis

Although in this simple numerical example, it may have been easier to go directly to this expression, in more complicated circuits sensitivity analysis is much easier

Example:



Ideally

$$I(s) = -\frac{1}{RCs} = -\frac{I_{0N}}{s}$$

$$(RC)^2\,\tau^2 I_{0A}^4 + I_{OA}^2\,(RC+\tau)^2 = 1$$

- Note that with the sensitivity analysis, it was not necessary to ever determine I_{0A} !!
- The sensitivity analysis was analytical, and only at the end was a numerical result obtained
- A parametric solution is usually necessary to compare different structures
- Though a closed-form analytical expression for I_{OA} could have been obtained for this simple circuit, closed-form solutions for parameters of interest often do not exist !
- Though the active sensitivity analysis was tedious, major simplifications for active sensitivity analysis will be discussed later.

How can sensitivity analysis be used to compare the performance of different circuits?

Circuits have many sensitivity functions

If two circuits have exactly the same number of sensitivity functions and all sensitivity functions in one circuit are lower than those in the other circuit, then the one with the lower sensitivities is a less sensitive circuit

But usually this does not happen !

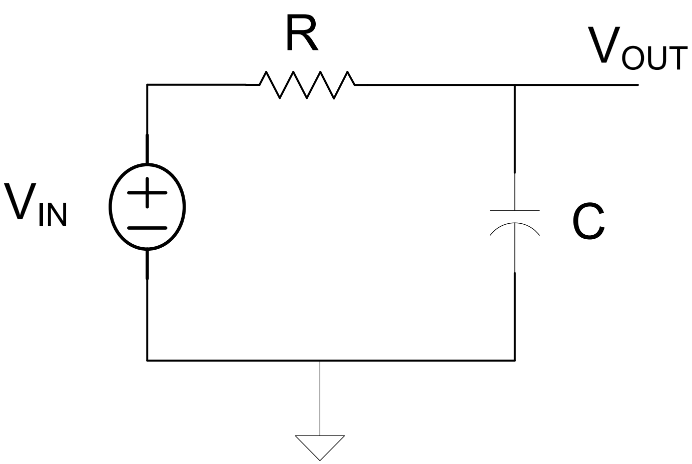
Designers would like a single metric for comparing two circuits !

$$\frac{dF}{F} = \sum_{i=1}^{k} \left(\boxed{S_{x_i}^{f} \Big|_{\vec{X}_N}} \bullet \boxed{\frac{dx_i}{x_{iN}}} \right)$$

Dependent on circuit structure
(for some circuits, also not dependent on components)

Dependent only on components
(not circuit structure)

Consider:

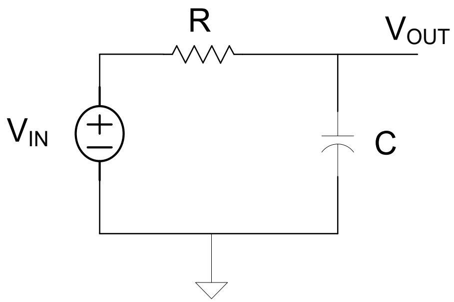


$$T(s) = \frac{1}{1+RCs}$$

$$T(s) = \frac{\omega_0}{s + \omega_0}$$

$$\omega_0 = \frac{1}{RC}$$

$$\omega_0 = \frac{1}{RC}$$



$$S_R^{\omega_0} = -1$$

$$S_C^{\omega_0} = -1$$

Dependent only on components
(not circuit structure)

$$\frac{d\omega_0}{\omega_0} = \sum_{i=1}^{2} \left(\boxed{S_{x_i}^{\omega_0} \Big|_{\vec{X}_N}} \bullet \boxed{\frac{dx_i}{x_{iN}}} \right)$$

$$\frac{d\omega_0}{\omega_0} = \boxed{[-1]} \bullet \boxed{\frac{dR}{R_N}} + \boxed{[-1]} \bullet \boxed{\frac{dC}{C_N}}$$

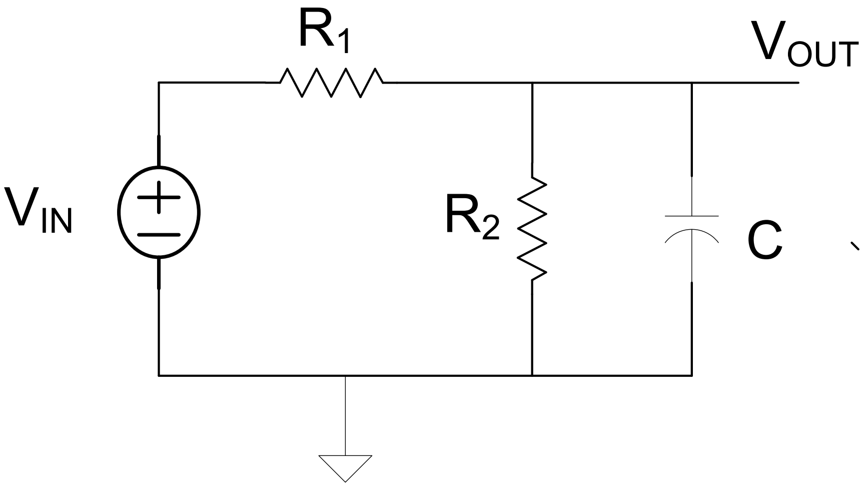
Dependent only on circuit structure

$$\frac{dF}{F} = \sum_{i=1}^{k} \left(\boxed{S_{x_i}^{f} \Big|_{\vec{X}_N}} \bullet \boxed{\frac{dx_i}{x_{iN}}} \right)$$

Dependent on circuit structure
(for some circuits, also not dependent on components)

Dependent only on components
(not circuit structure)

Consider now:



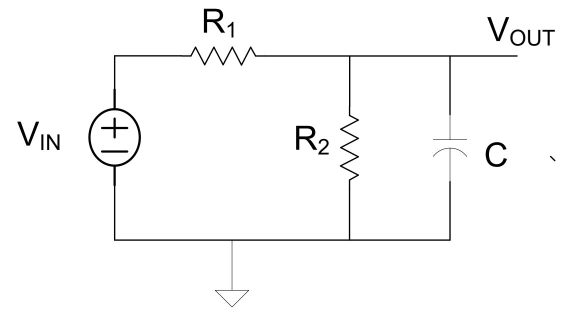
$$T(s) = \frac{\frac{R_2}{R_1+R_2}}{1+\left(\frac{R_1R_2}{R_1+R_2}C\right)s}$$

$$T(s) = \frac{R_2}{R_1+R_2} \bullet \frac{\omega_0}{s+\omega_0}$$

$$\omega_0 = \frac{R_1+R_2}{R_1R_2C}$$

$$S_{R_1}^{\omega_0} = ?$$

$$\omega_0 = \frac{R_1+R_2}{R_1R_2C}$$



$$\omega_0 = \frac{G_1+G_2}{C}$$

$$S_{R_1}^{\omega_0} = -S_{G_1}^{\omega_0}$$

$$S_{G_1}^{\omega_0} = S_{G_1}^{G_1+G_2}$$

$$S_{G_1}^{G_1+G_2} = \left(\frac{\partial\left(G_1+G_2\right)}{\partial G_1}\right)\frac{G_1}{G_1+G_2} = \frac{G_1}{G_1+G_2}$$

$$S_{R_1}^{\omega_0} = -\frac{R_2}{R_1+R_2}$$

Note this is dependent upon the components as well !
Actually dependent upon component ratio!

Theorem: If $f(x_1, ..x_m)$ can be expressed as $f = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$

where $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$ are real numbers, then $S_{x_i}^{f}$ is not dependent upon any of the variables in the set $\{x_1, ..x_m\}$

Proof:

$$S_{x_i}^{f} = S_{x_i}^{X_i^{\alpha_i}}$$

$$S_{x_i}^{f} = \alpha_i$$

$$S_{x_i}^{X_i^{\alpha_i}} = \frac{\partial X_i^{\alpha_i}}{\partial x_i} \bullet \frac{x_i}{X_i^{\alpha_i}}$$

$$S_{x_i}^{X_i^{\alpha_i}} = \alpha_i X_i^{\alpha_i - 1} \bullet \frac{x_i}{X_i^{\alpha_i}}$$

It is often the case that functions of interest are of the form expressed in the hypothesis of the theorem, and in these cases the previous claim is correct

$$S_{x_i}^{X_i^{\alpha_i}} = \alpha_i$$

Theorem: If $f(x_1, ..x_m)$ can be expressed as $f = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$

where $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$ are real numbers, then the sensitivity terms in

$$\frac{df}{f} = \sum_{i=1}^{k} \left(S_{x_i}^{f} \Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}} \right)$$

are dependent only upon the circuit architecture and not dependent upon the components and and the right terms are dependent only upon the components and not dependent upon the architecture

This observation is useful for comparing the performance of two or more circuits where the function f shares this property

Metrics for Comparing Circuits

Summed Sensitivity

$$\rho_S = \sum_{i=1}^{m} S_{x_i}^{f}$$

Not very useful because sum can be small even when individual sensitivities are large

Schoeffler Sensitivity

$$\rho = \sum_{i=1}^{m} \left| S_{x_i}^{f} \right|$$

Strictly heuristic but does differentiate circuits with low sensitivities from those with high sensitivities

Metrics for Comparing Circuits

$$\rho = \sum_{i=1}^{m} \left| S_{x_i}^{f} \right|$$

Often will consider several distinct sensitivity functions to consider effects of different components

$$\rho_R = \sum_{All\ resistors} \left| S_{R_i}^{f} \right|$$

$$\rho_C = \sum_{All\ capacitors} \left| S_{C_i}^{f} \right|$$

$$\rho_{OA} = \sum_{All\ op\ amps} \left| \mathcal{S}_{\tau_i}^{f} \right|$$

Homogeniety (defn)

A function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ if

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

Note: f may be comprised of more than n variables

Theorem: If a function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ then

$$\sum_{i=1}^{n} S_{x_i}^{f} = m$$

Proof:

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

Differentiate WRT λ

$$\frac{\partial\left(f(\lambda x_1, \lambda x_2, \ldots \lambda x_n)\right)}{\partial \lambda} = m\lambda^{m-1} f(x_1, x_2, \ldots x_n)$$

$$\frac{\partial f}{\partial \lambda x_1} x_1 + \frac{\partial f}{\partial \lambda x_2} x_2 + \ldots + \frac{\partial f}{\partial \lambda x_n} x_n = m\lambda^{m-1} f(x_1, x_2, \ldots x_n)$$

$$\frac{\partial f}{\partial \lambda x_1} x_1 + \frac{\partial f}{\wp \lambda x_2} x_2 + \ldots + \frac{\partial f}{\partial \lambda x_n} x_n = m\lambda^{m-1} f\left(x_1, x_2, \ldots x_n\right)$$

Simplify notation

$$\frac{\partial f}{\partial \lambda x_1} x_1 + \frac{\partial f}{\partial \lambda x_2} x_2 + \ldots + \frac{\partial f}{\partial \lambda x_n} x_n = m\lambda^{m} f$$

Divide by f

$$\frac{\partial f}{\lambda x_1} \frac{x_1}{f} + \frac{\partial f}{\lambda x_2} \frac{x_2}{f} + \ldots + \frac{\partial f}{\lambda x_n} \frac{x_n}{f} = m\lambda^{m}$$

Since true for all λ, also true for λ=1, thus

$$\frac{\partial f}{x_1} \frac{x_1}{f} + \frac{\partial f}{x_2} \frac{x_2}{f} + \ldots + \frac{\partial f}{x_n} \frac{x_n}{f} = m$$

This can be expressed as

$$\sum_{i=1}^{n} S_{x_i}^{f} = m$$

Theorem: If a function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ then

$$\sum_{i=1}^{n} S_{x_i}^{f} = m$$

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

The concept of homogeneity and this theorem were somewhat late to appear

Are there really any useful applications of this rather odd observation?

Let T(s) be a voltage or current transfer function

(i.e. dimensionless)

Observation: Impedance scaling does not change any of the following, provided Op Amps are ideal:

$$T(s),\ T(j\omega),\ |T(j\omega)|,\ \omega_0,\ Q,\ p_k,\ z_k$$

So, consider impedance scaling by a parameter λ

$$R \to \lambda R$$

$$L \to \lambda L$$

$$C \to C/\lambda$$

For these <u>impedance</u> <u>invariant</u> functions

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^0 f(x_1, x_2, \ldots x_n)$$

Thus, all of these functions are homogeneous of order m=0 in the impedances

Let T(s) be a Transresistance or Transconductance Transfer Function

Observation: Impedance scaling does not change any of the following, provided Op Amps are ideal:

ω_0, Q, p_k, z_k, band edge

(these are impedance invariant functions)

So, consider impedance scaling by a parameter λ

$$R \rightarrow \lambda R$$
$$L \rightarrow \lambda L$$
$$C \rightarrow C/\lambda$$

For these impedance invariant functions

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^0 f(x_1, x_2, \ldots x_n)$$

Thus, all of these functions are homogeneous of order m=0 in the impedances

Theorem 1: If all op amps in a filter are ideal, then ω_o, Q, BW, all band edges, and all poles and zeros are homogeneous of order 0 in the impedances.

Theorem 2: If all op amps in a filter are ideal and if T(s) is a dimensionless transfer function, $T(s)$, $T(j\omega)$, $|T(j\omega)|$, $\angle T\left(j\omega\right)$, are homogeneous of order 0 in the impedances

Theorem 1: If all op amps in a filter are ideal, then ω_o, Q, BW, all band edges, and all poles and zeros are homogeneous of order 0 in the impedances.

Proof of Theorem 1

These functions are all impedance invariant so if follows trivially that they are homogeneous of order 0 in all of the impedances

Theorem 3: If all op amps in a filter are ideal and if T(s) is an impedance transfer function, T(s) and T(jω) are homogeneous of order 1 in the impedances

Theorem 4: If all op amps in a filter are ideal and if T(s) is a conductance transfer function, T(s) and T(jω) are homogeneous of order -1 in the impedances

Corollary 1: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if a function f is homogeneous of order 0 in the impedances, then

$$\sum_{i=1}^{k_1} S_{R_i}^{f} = \sum_{i=1}^{k_2} S_{C_i}^{f}$$

Corollary 2: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors then

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0$$

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$$

Corollary 1: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if a function f is homogeneous of order 0 in the impedances, then

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Proof of Corollary 1:

Corollary 1: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if a function f is homogeneous of order 0 in the impedances, then

$$\sum_{i=1}^{k_1} S_{R_i}^{f} = \sum_{i=1}^{k_2} S_{C_i}^{f}$$

Proof:

Since f is homogenous of order zero in the impedances, z_1, z_2, … z_{k1+k2},

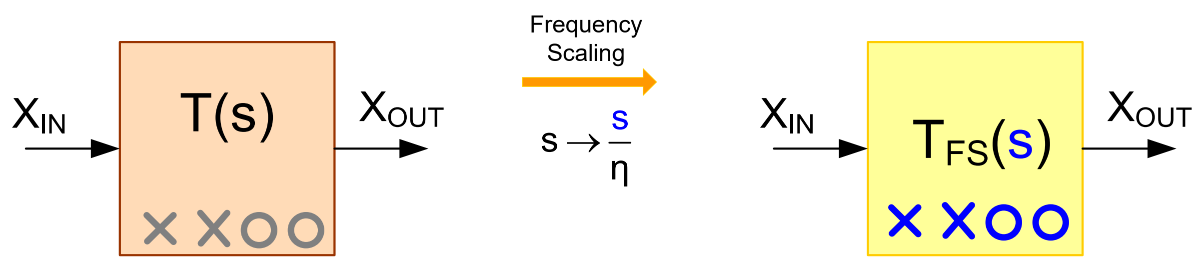
$$\sum_{i=1}^{k_1+k_2} S_{z_i}^{f} = 0$$

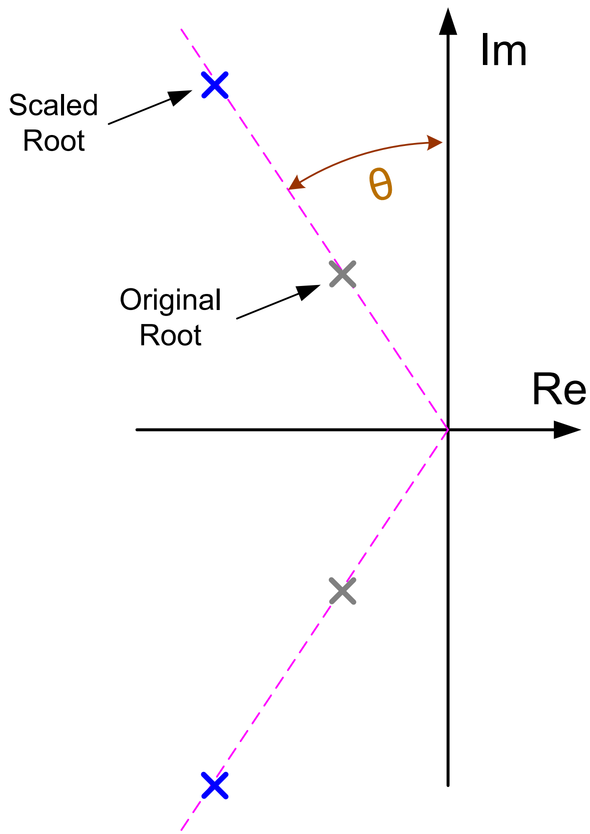
$$\therefore \quad \sum_{i=1}^{k_1} S_{R_i}^{f} + \sum_{i=1}^{k_2} S_{1/C_i}^{f} = 0$$

$$\therefore \quad \sum_{i=1}^{k_1} S_{R_i}^{f} - \sum_{i=1}^{k_2} S_{C_i}^{f} = 0$$

Proof of Corollary 2:

Recall:





Frequency Scaling: Scaling all frequency-dependent elements by a constant

$$L \rightarrow \eta L$$

$$C \rightarrow \eta C$$

Theorem: If all components are frequency scaled, roots (poles and zeros) will move along a constant Q locus

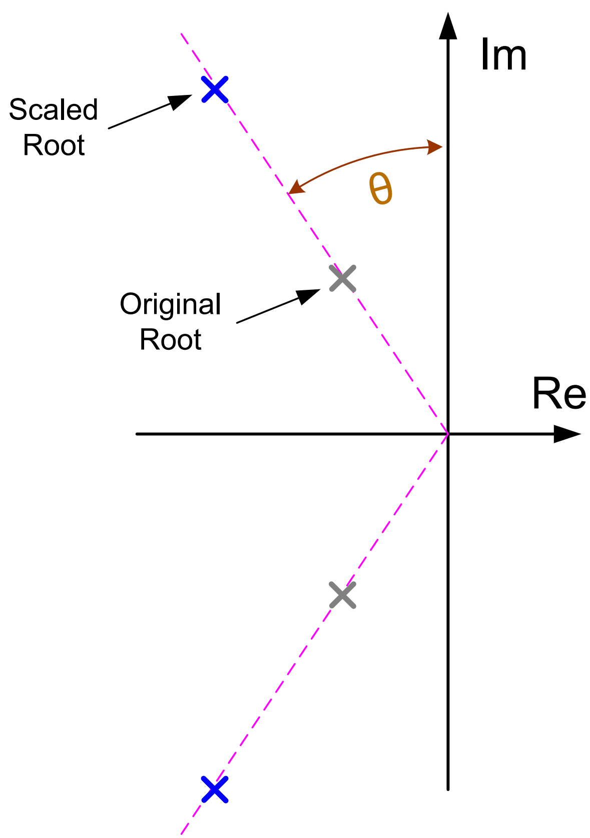
Proof of Theorem:

$$T_{FS}(s) = T(s)\Big|_{s=\frac{s}{\eta}}$$

Proof of Corollary 2:

Recall:

Theorem: If all components are frequency scaled, roots (poles and zeros) will move along a constant Q locus



Proof: $$T_{FS}(s) = T(s)\Big|_{s=\frac{s}{\eta}}$$

Let p be a pole (or zero) of T(s)

$T(p)=0$ consider $\quad p = \dfrac{p}{\eta}$

$$T_{FS}(s) = T\left(\frac{s}{\eta}\right) = T(s)$$

Since true for any variable, substitute in p

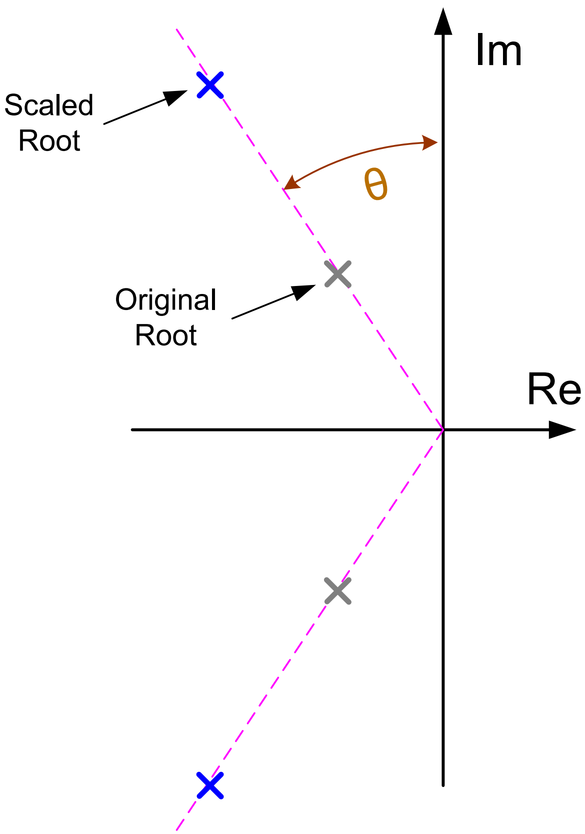
$$T_{FS}(p) = T\left(\frac{p}{\eta}\right) = T(p) = 0$$

Thus **p** is a pole (or zero) of $T_{FS}(s)$

Proof of Corollary 2:

Recall:

Theorem: If all components are frequency scaled, roots (poles and zeros) will move along a constant Q locus



Proof: Thus **p** is a pole (or zero) of $T_{FS}(s)$

$$p = \frac{p}{\eta}$$

$$p = p\eta$$

Express **p** in polar form

$$p = re^{j\beta}$$

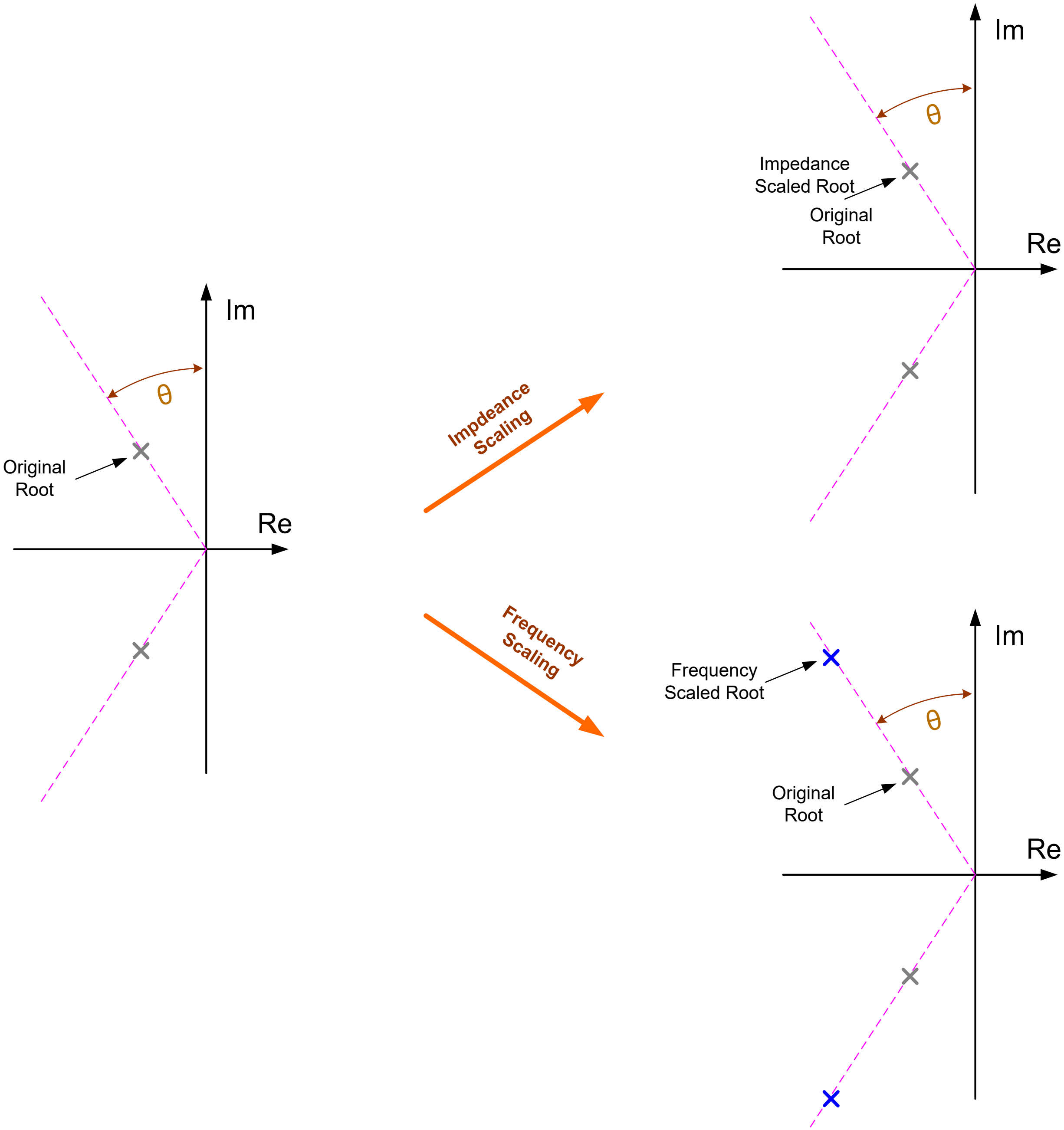
$$p = \eta p = \eta r e^{j\beta}$$

Thus **p** and **p** have the same angle

Thus the scaled root has the same root Q

Proof of Corollary 2:   Impedance and Frequency Scaling

Recall:



Proof of Corollary 2:

Corollary 2: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors then $\sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$ and $\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0$

Since impedance scaling does not change pole (or zero) Q, the pole (or zero) Q must be homogeneous of order 0 in the impedances

(For more generality, assume k_3 inductors)

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} + \sum_{i=1}^{k_2} S_{1/C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (1)$$

Since frequency scaling does not change pole (or zero) Q, the pole (or zero) Q must be homogeneous of order 0 in the frequency scaling elements

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (2)$$

Proof of Corollary 2:

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} + \sum_{i=1}^{k_2} S_{1/C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (1)$$

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (2)$$

From theorem about sensitivity of reciprocals, can write (1) as

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} - \sum_{i=1}^{k_2} S_{C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (3)$$

It follows from (2) and (3) that

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} - 2\sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (4)$$

Since RC network, it follows from (4) and (2) that

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0 \qquad \sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$$